International Journal of Theoretical Physics, Vol. 23, No. 1, 1984

Sheaf Models and Massless Fields[1]

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Received May 3, 1983

We give an account of the logical and model theoretic aspects of sheaf theory and describe how this formalism leads to a new interpretation of the role of sheaves in the twistor description of massless fields.

1. INTRODUCTION

The characterization of massless fields on complexified Minkowski space, in terms of the sheaf cohomology of twistor space, is described in Eastwood, Penrose, Wells (1981), which also contains an extensive bibliography on the subject. The use of sheaf theory in the description is novel among the techniques of mathematical physics and arises from a nonlocal representation of physical properties on twistor space, in contrast to the conventional description in terms of local differential equations on space-time. This departure is inspired to a certain extent by problems with the modeling of the space-time continuum. The usual model based on the real line has a great deal of local structure, including a structure of points which can be resolved with infinite precision. This model resulted from idealizing observations in classical physics. More recent developments in quantum mechanics, field theories, and attempts at quantizing gravity encounter difficulties which seem to be related to this classically idealized small-scale structure and thus it is of great interest to consider theories which represent the continuum in an essentially different way.

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0020-7748/84/0100-0067$03.50/0

Sheaf theory was originally developed in the context of algebraic geometry as a tool for passing from local to global properties of spaces, and it is this aspect of sheaves that is used in the work of Eastwood, Penrose, Wells (1981). More recently it was discovered that sheaves also have important applications in logic and model theory: the category of all sheaves on a fixed topological space can be viewed as a model for intuitionistic set theory. This aspect of sheaves was first developed by F. W. Lawvere (1975, 1976) in the early 1970s and a comprehensive account may be found in the book by P. Johnstone (1977). It is the purpose of this work to investigate the role of sheaves in twistor theory from this set theoretic point of view. One of the most interesting aspects of the sheaf model formalism is that it provides a framework in which a sheaf can be viewed as a model of a space which does not have a local structure of points but a structure which is "intrinsically continuous" in a precise mathematical sense (cf. Lawvere, 1975, 1976). It is hoped that this mathematical development may shed light on the physical basis of twistor theory and its unusual representation of the continuum.

Let X be a (fixed) topological space and $\mathrm{Op}(X)$ the lattice of its open sets ordered by inclusions.

Definition 1.1. A sheaf S of sets over X is an assignment of a set $\mathsf{S}(U)$ to each nonempty open set $U \subseteq X$ and a map

$$\mathsf{S}(U) \xrightarrow{\rho_{UV}} \mathsf{S}(V)$$

whenever $U \supseteq V$. This structure is required to satisfy the following:

(i) If $U \subseteq V \subseteq W$ when $\mathsf{S}(W) \xrightarrow{\rho_{WU}} \mathsf{S}(U)$ is the same map as the composite

$$\mathsf{S}(W) \xrightarrow{\rho_{WV}} \mathsf{S}(V) \xrightarrow{\rho_{VU}} \mathsf{S}(U)$$

(ii) Let $\{U_i\}$ be any open cover of U and let $U_{ij} = U_i \cap U_j$ be the intersections. Let $\sigma_i \in \mathsf{S}(U_i)$ be a collection of elements such that

$$\rho_{U_iU_{ij}}(\sigma_i) = \rho_{U_jU_{ij}}(\sigma_j) \qquad \text{for all } i, j$$

Then there is a unique $\sigma \in \mathsf{S}(U)$ with $\sigma_i = \rho_{UU_i}(\sigma)$

We often refer to the elements $\sigma \in \mathsf{S}(U)$ as "the elements of S over U" or "the sections of S over U." ρ_{UV} is called the restriction map from U to the smaller open set V. Condition (ii) then states that if we have a collection of sections of S over an open cover of U, which agree when restricted to the overlaps, then they can be patched together in a unique way to give a single section over U.

We will view a sheaf S as a set whose elements are all the sections $\sigma \in \mathsf{S}(U)$ over all the open sets. The difference from the usual notion of a set is that here we have an extra internal structure of a grading of the elements by the lattice $\mathrm{Op}(X)$. Heuristically we think of the open sets as measures of definability and the elements in $\mathsf{S}(U)$ are the members of S defined to the extent U. The larger the open set, the greater the definability so that the completely defined elements are those in $\mathsf{S}(X)$, i.e., the global sections. The restriction maps ρ_{UV} can then be thought of as relating an element $\sigma \in \mathsf{S}(U)$ to the same element with part of its definability ignored, hence having a weaker measure $V \subset U$ of definability. Condition (ii) in Definition 1.1 may be interpreted as allowing the construction or definition of elements by consolidating weaker but compatible definitions.[3]

It is natural to require that maps between these structures preserve the $\mathrm{Op}(X)$ grading of elements:

Definition 1.2. Let S and T be sheaves over X. A sheaf map φ: $\mathsf{S} \to \mathsf{T}$ is a collection of set maps $\varphi(U)$: $\mathsf{S}(U) \to \mathsf{T}(U)$ such that whenever $V \subseteq U$, the following diagram commutes:

$$\begin{tikzcd} \mathsf{S}(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho_{UV}"'] & \mathsf{T}(U) \arrow[d, "\rho_{UV}"] \\ \mathsf{S}(V) \arrow[r, "\varphi(V)"'] & \mathsf{T}(V) \end{tikzcd}$$

We denote by $\mathrm{Sh}(X)$ the category of all sheaves and sheaf maps over X. It is shown in Sections 2 and 3 that $\mathrm{Sh}(X)$ has a rich enough internal structure to allow analogs of all the familiar constructions of mathematics which we usually carry out with sets, e.g., the formation of products, unions, construction of the set of all maps between two sets, the set of all subsets of a set. This means that in $\mathrm{Sh}(X)$ we can develop analogs of most familiar mathematical definitions and theories like the notions of topology, vector spaces, partial differentiation, etc.

The category of sets appears as a special case of the sheaf formalism obtained by taking X as the one-element topological space. From an objective mathematical point of view, all the universes, $\mathrm{Sh}(X)$, are equally good "arenas for mathematical constructions" but historically only the use

[3] The $\mathrm{Op}(X)$ grading has also been given other intuitive interpretations and derives originally from the semantics of intuitionistic logic developed by E. Beth (1947) and S. Kripke (1965).

of sets was recognized. In the category of sets every object is fully characterized by its (completely defined) point elements. For mathematical physics we may not want this feature (particularly in the representation of the continuum) and thus be lead to consider some other category Sh(X) with X determined by further physical information about the small-scale structure of the continuum. In Sh(X) the pointlike elements of a sheaf S are its global sections and a general sheaf is far from being characterized by these, possibly having no global sections at all.

For later reference, we will give a statement of the main twistorial results about massless fields. A complete account is given in Eastwood, Penrose, Wells (1981) or in Wells (1979).

Definition 1.3. (i) A massless field (on an open subset of complexified Minkowski space) of helicity $n/2$ ($n=1,2,\ldots$) is a holomorphic symmetric spinor field $\varphi^{A'\ldots L'}$ with n indices satisfying

$$\nabla_{AA'}\varphi^{A'\ldots L'}=0$$

(ii) A massless field of helicity $-n/2$ ($n=1,2,\ldots$) is a holomorphic spinor field $\varphi_{AB\ldots L[B'X']\ldots[L'Z']}$ with n symmetric unprimed indices and $(n-1)$ pairs of skew primed indices satisfying

$$\nabla_{P'[P}\varphi_{A]B\ldots L[B'X']\ldots[L'Z']}=0$$

(iii) A potential for an $-n/2$ helicity massless field is a spinor field $\psi_{AB'\ldots L'}$ with an unprimed index and $(n-1)$ symmetric primed indices satisfying

$$\nabla_{[P(P'}\psi_{A]B')\ldots L'}=0$$

(here we skew over PA and symmetrize over $P'B'$).

Usually the one-index spin spaces are two dimensional and hence the spaces with a pair of skewed indices are one dimensional (spanned by a skew spinor normally denoted ε). Thus, defining $\nabla^{AA'}$ as $\varepsilon^{A'B'}\varepsilon^{AB}\nabla_{BB'}$ the definition 1.3 (ii) of negative helicity massless fields becomes equivalent to the more usual

$$\nabla^{AA'}\varphi_{A\ldots L}=0$$

However, later we will relax the condition of two dimensionality of the primed spin space and it will be necessary to use the more cumbersome definition 1.3 (ii) which we adopt here at the outset.

Notation 1.4. Let $\mathcal{S}^{\cdots}_{\cdots}$ denote the spin space with a general index structure. We define the following sheaves on complexified Minkowski space:

$\mathsf{S}^{\cdots}_{\cdots}$	sheaf of holomorphic $\mathcal{S}^{\cdots}_{\cdots}$ valued functions
$\mathsf{Z}^{A'\ldots L'}$	sheaf of positive helicity massless fields
$\mathsf{Z}_{A\ldots L[B'X']\ldots[L'Z']}$	sheaf of negative helicity massless fields
$\mathsf{P}_{AB'\ldots L'}$	sheaf of potentials for negative helicity fields
$\mathsf{T}_{A'\ldots L'}$	sheaf of holomorphic symmetric fields $\varphi_{A'\ldots L'}$ satisfying $\nabla_{X(X'}\varphi_{A'\ldots L')} = 0$.

The basic structural space-time properties of massless fields are expressed in the form of two exact sequences, the first being a resolution of the differential operator appearing in the massless field equations and the second is the expression of massless fields as potentials modulo a gauge freedom.

Theorem 1.5. The following sequences are exact:

$$\text{(a)}\quad 0 \to \mathsf{Z}^{A'\ldots L'} \hookrightarrow \mathsf{S}^{(A'\ldots L')} \xrightarrow{\nabla_{AA'}} \mathsf{S}_A^{(B'\ldots L')} \xrightarrow{\nabla_{B'[B}} \mathsf{S}_{[AB]}^{(C'\ldots L')} \to 0$$

$$\text{(b)}\quad 0 \to \mathsf{T}_{C'\ldots L'} \hookrightarrow \mathsf{S}_{(C'\ldots L')} \xrightarrow{\nabla_{A(B'}} \mathsf{P}_{AB'\ldots L'} \xrightarrow{\nabla_{(B[X'}\cdots\nabla_{(L[Z'}}$$
$$\to \mathsf{Z}_{A\ldots L[B'X']\ldots[L'Z']} \to 0$$

[Here, there are $n-2$ indices $C'\ldots L'$ and the map $\mathsf{P} \to \mathsf{Z}$ is the application of $(n-1)$ ∇ operations followed by symmetrization over all unprimed indices and skew symmetrization over pairs of primed indices as indicated on the index structure of Z.]

Turning now to the twistor description of massless fields, let PT^+ be the part of projective twistor space having positive twistor norm. We denote by $\mathsf{T}(n)$ the sheaf of holomorphic twistor functions homogeneous of degree n. Let M^+ denote the future tube of complexified Minkowski space.

Theorem 1.6. For any n, there is a natural one to one correspondence between (i) helicity $n/2$ massless fields on M^+ and (ii) the sheaf cohomology group $H^1(PT^+, \mathsf{T}(-n-2))$.

Let $F^+ = M^+ \times \mathcal{S}_{A'}$ be the primed spin bundle on M^+ and, following the usual twistor notation, introduce coordinates $(x^{AA'}, \pi_{A'})$ on F^+. The sheaf $\mathsf{T}(n)$ may be identified with the sheaf [also denoted $\mathsf{T}(n)$] on F^+ of functions $f(x^{AA'}, \pi_{A'})$ homogeneous of degree n in $\pi_{A'}$ and satisfying $\pi^{A'}\nabla_{AA'}f = 0$. Theorem 1.6 may then be derived by developing the cohomology of the following exact sequence on F^+.

Sequence 1.7:

$$0 \to \mathsf{T}(n) \hookrightarrow \mathsf{F}(n) \overset{\pi^{A'}\nabla_{AA'}}{\to} \mathsf{F}_A(n+1) \overset{\pi^{A'}\nabla^A_{A'}}{\to} \mathsf{F}(n+2) \to 0$$

Here $\mathsf{F}(n)$ is the sheaf of all holomorphic functions on F^+ homogeneous of degree n in $\pi_{A'}$, and $\mathsf{F}_A(n)$ is the sheaf of $\mathcal{S}_A$ valued functions homogeneous of degree n in $\pi_{A'}$.

The main point of this paper is to show that the sequences 1.5 (a) and (b) on the one hand (giving the standard space-time description of massless fields) and the sequence 1.7 on the other hand (leading to the twistor description) are two different models of the *same* mathematical theory interpreted in two different categories of sheaves (apart from some minor modifications). In other words, the novelty of the twistor description is attributed here, not to changing our description of massless fields, but rather to working with the same theory of massless fields in a more exotic category than the category of sets.

In Section 2 we set down the basics of model theory distinguishing carefully between the notion of a theory and the notion of its models obtained by interpreting the axioms of the theory in concrete mathematical structures according to precise given rules of interpretation. In Section 3 we describe explicitly the rules of interpretation and satisfaction for sheaf models. Sections 2 and 3 together are meant to provide an essentially self-contained exposition of the theory of sheaf models, since this is unfamiliar in the literature of mathematical physics. Section 4 describes some examples of mathematical constructions carried out in sheaf models which are required later for the interpretation of the theory of massless fields. In Section 5 we set up a language and axioms for the theory of massless fields, in a form suitable for interpretation in sheaf categories. Finally Section 6 describes a sheaf model of this theory and it is shown that this model is closely related to the twistor description of massless fields.

2. MODEL THEORY

It will be of central importance to distinguish between a *theory* (axioms, statements) and *models* of a theory (actual mathematical structures, e.g., sets and set maps in which the axioms are interpreted). We clarify this with a familiar example. The notion of a group (G, μ, τ, e) is defined axiomatically by the following statements:

$$\forall x \forall y \forall z(\mu(x, \mu(y, z)) = \mu(\mu(x, y), z))$$

$$\forall x(\mu(x, \tau(x)) = \mu(\tau(x), x) = e)$$

$$\forall x(\mu(x, e) = \mu(e, x) = x)$$

Here μ is the multiplication and τ is a function giving inverses. The theory of groups is the collection of all statements that can be derived from these axioms using the laws of logic. (Here, a law of logic is simply a prescription for constructing a new true statement from given statements which are already regarded as being true). Thus, a theory consists of formal statements written in a language and makes no reference to concrete mathematical structures.

A model for the theory of groups (in the category of sets) is a set $\tilde{G}$ and maps $\tilde{\mu}$: $\tilde{G} \times \tilde{G} \to \tilde{G}$, $\tilde{\tau}$: $\tilde{G} \to \tilde{G}$ and a selected element $\tilde{e} \in \tilde{G}$ which satisfy the above axioms. This notion of model requires rules of interpretation, specifying how to associate concrete mathematical objects to the symbols of the language, and also rules of satisfaction specifying when a statement written in the language is considered to hold, or be true, in a given interpretational structure. Thus, in the above example, the symbol μ in the language is interpreted by the set map $\tilde{\mu}$ and any such interpretation satisfies the associative law if certain set theoretic constructions involving $\tilde{G}$ and $\tilde{\mu}$ hold.

These notions of theories and models embody a widely used basis for mathematics but this is not the only possible foundation. It is not clear, for example, that a purely geometrical approach or a constructivist approach, can be forced into this mold. Both of these perhaps have a more Platonistic flavor, in which ideal mathematical objects exist, and there is no doubt about what the "correct" model of any notion is, so the distinction between the model and the theory is unnecessary.

In constructing mathematical models in physics the distinction between theories and their models is rarely made even though the physical content is often abstracted in terms of statements of properties, i.e., a theory. In general, many inequivalent models exist, (even just in the category of sets) and the particular one chosen is determined by constructs of existing physical theories and partly by historical precedents. The choice of some particular model is a strong assumption (often unacknowledged) in the theory since it can largely determine its mathematical form.

Perhaps the most interesting example of this is the use of the set of real numbers to model the outcomes of measurements (e.g., of reading a pointer on a scale). A realistic analysis of the properties of these outcomes, taking into account accuracy and definability, is very complex and the real number system satisfies most requirements only in an idealized way. However, even in the category of sets, there are many other non-isomorphic models also satisfying all these requirements. From a mathematical viewpoint, a real number is an infinitely complex object and has many extra special properties (especially on the small scale) which cannot be justified from physics. These extra nonphysical properties (which differ for different models) play an important role in the mathematical development of the physical theory and hence the choice of a particular model is important.

The mathematical discipline of model theory is a useful tool for investigating the dependence of a physical theory and its predictions on the choice of particular mathematical models and also for investigating the variety of available models. To state the basic model theoretic rules of interpretation and satisfaction it is necessary to be precise about the language used to formulate the axioms. For our purposes it will be sufficient to use a first-order, many sorted language.

Definition 2.1. A first-order many sorted language L is a collection of symbols of the kind shown in Table I. Using these symbols we construct

TABLE I

Symbol	Explanatory Remarks
A collection of symbols s_i called sorts	These will denote (domains of) different kinds of variables
A collection of free variables x_j for each sort s	These will be used for building up expressions.
Sorted relation symbols, R	$R(x_1, \ldots, x_n)$ will denote a relation between the variables of the designated sorts.
Sorted function symbols, f	$f(x_1, \ldots, x_n)$ of sort s denotes a function of the variables of the designated sorts taking values in sorts s.
Individual constant symbols of sort s	These denote special named elements of sort s.
The logical symbols $\vee$ (disjunction) & (conjunction) $\neg$ (negation) $\Rightarrow$ (implication) $\exists$ (existential quantifier) $\forall$ (universal quantifier) (,) (brackets)	These are used for constructing statements.

(inductively) the following.

Terms: All free variable symbols and constant symbols are terms. If $f(x_1,\ldots,x_n)$ is a function symbol of sort s and $t_1,\ldots,t_n$ are terms of the sorts of $x_1,\ldots,x_n$, then $f(t_1,\ldots,t_n)$ is a term of sort s. All terms are obtained in this way.

Formulae: Atomic formulas are expressions of the form $R(t_1,\ldots,t_n)$ where R is a relation symbol and $t_1,\ldots,t_n$ are terms of the appropriate sorts. General formulas of greater complexity, are built up from atomic formulas using the logical symbols in the familiar ways [the rules are written out in Chang and Keisler (1973)].

In the example of groups, G is a sort symbol, $=$ a two-place relation, and the multiplication μ is a two-place function symbol of sort G. e is a constant symbol and τ a function symbol, both of sort G.

We include some rules for constructing new sorts out of given ones. Given sorts s_1, s_2 we can construct the product sort $s_1 \times s_2$. A formula $\varphi(x_1, x_2)$ with variables x_1, x_2 of sorts s_1, s_2, respectively, may then be regarded as $\varphi(x_3)$, where x_3 has sort $s_1 \times s_2$. We also allow the construction of "subsorts": if $\varphi(x)$ is a formula, x of sort s, then we can construct a new sort denoted $\{x|\varphi(x)\}$ which represents the domain of values x of sort s for which $\varphi(x)$ holds. If s_1 and s_2 are sorts we have the construction of the function space sort denoted $s_2^{s_1}$, representing the sort of all maps from s_1 to s_2. To express their intended meanings, these constructions are normally subjected to various axioms [which may be found in Johnstone (1977)]. Sometimes it is useful in quantified formulas to make the sort of a variable explicit. We will use the following notation: the formula $\forall x\varphi(x)$ with x of sort s may be written as $(\forall x \in s)(\varphi(x))$. "$\forall x \in s$" is read as "for all x of sort s"; similarly for $\exists x \in s$.

An interpretation M of the language L in the category of sets is an assignment, Table II. The last clause in this table gives the interpretation of atomic formulas. Nonatomic formulas are built up out of these as follows. Let $\varphi(x_1,\ldots,x_n)$ and $\psi(x_1,\ldots,x_n)$ be formulas with free variables of sorts $s_1,\ldots,s_n$ so $M(\varphi)$ and $M(\psi)$ are subsets of $M(s_1)\times \cdots \times M(s_n)$. Then we set

$$M(\varphi \& \psi) = M(\varphi) \cap M(\psi)$$

$$M(\varphi \vee \psi) = M(\varphi) \cup M(\psi)$$

$$M(\varphi \Rightarrow \psi) = M(\psi) \cup \overline{M(\varphi)} \qquad (\overline{} \text{ denotes complement})$$

$$M(\neg\varphi) = \overline{M(\varphi)}$$

$$M(\exists x_1 \varphi(x_1,\ldots,x_n)) = \{(a_2,\ldots,a_n) \in M(s_2) \times \cdots \times M(s_n) | \text{ there is an } a_1 \in M(s_1) \text{ with } (a_1, a_2,\ldots,a_n) \in M(\varphi)\}$$

$$M(\forall x_1 \varphi(x_1,\ldots,x_n)) = \{(a_2,\ldots,a_n) \in M(s_2) \times \cdots \times M(s_n) | \text{ for all } a_1 \in M(s_1)\ (a_1, a_2,\ldots,a_n) \in M(\varphi)\}$$

Using these rules we can construct the interpretation of any formula or term. Note that if φ has no free variables (i.e., is a proposition or sentence) then $M(\varphi)$ is a subset of the product of no factors which can be identified as the one-element set 1. 1 has exactly two subsets which are conventionally called "true" and "false". The above interpretations of &, $\vee, \neg$ on these truth values are the usual Boolean algebra operations on the subsets of 1.

The interpretation of the product sort $s_1 \times s_2$ is simply the Cartesian set product $M(s_1) \times M(s_2)$ and for x of sort s, $\{x | \varphi(x)\}$ is interpreted by the

TABLE II

Symbol	Interpretation M
A sort s	A set $M(s)$ (the domain of sort s)
A relation symbol $R(x_1,\ldots,x_n)$ (x_i of sort s_i)	A subset $M(R)$ of $M(s_1) \times \cdots \times M(s_n)$ (i.e., the n tuples of elements for which the relation holds)
A function symbol $f(x_1,\ldots,x_n)$ of sort s (x_i of sort s_i)	A set map $M(f)$: $M(s_1) \times \cdots \times M(s_n) \to M(s)$
An individual constant c of sort s	An element $M(c) \in M(s)$, i.e., a map $1 \to M(s)$
A variable x of sort s	The identity map $M(s) \to M(s)$
Then, inductively, for terms $t_1,\ldots,t_n$ of sorts $s_1,\ldots,s_n$ with free variables among sorts $s_1,\ldots,s_j$:	
$f(t_1,\ldots,t_n)$ of sort s	$M(f(t_1,\ldots,t_n))$: $M(s_1) \times \cdots \times M(s_j) \to M(s)$ obtained by composing the interpretations of the terms t_i with the interpretation of f
$R(t_1,\ldots,t_n)$	$M(R(t_1,\ldots,t_n))$ is the subset of $M(s_1) \times \cdots \times M(s_j)$ of all j-tuples such that the relation R holds between the terms $t_1,\ldots,t_n$ evaluated on the j-tuple

subset $M(\varphi) \subseteq M(s)$. $M(s_s^{s_1})$ is the set $M(s_2)^{M(s_1)}$ of all set maps from $M(s_1)$ to $M(s_2)$. It is fortuitous that the collection of all maps between $M(s_1)$ and $M(s_2)$ is again a set. Later, for sheaves, the collection of all sheaf maps between two sheaves is also a set, and not a sheaf, so the interpretation of function space sorts is less obvious there.

Finally, the rule of satisfaction is simply that "the formula $\varphi(x_1, \ldots, x_n)$ is true under the interpretation M" or "M is a model of φ" if $M(\varphi) \subseteq M(s_1 \times \cdots \times s_n)$ is the whole of $M(s_1 \times \cdots \times s_n)$.

These rules are so natural that outside treatises on model theory they are rarely stated, although used in practically every textbook. We state them here because the rules of interpretation in sheaf categories (given in the next section) which look more complicated, are, in fact, direct generalizations of the above rules.

3. SHEAF MODELS

The category Sh(X) of sheaves and sheaf maps over a topological space X has been defined in Section 1. Before listing the interpretation rules in this category, we need to describe the sheaf analogs of certain constructions with sets. The product $\mathsf{S} \times \mathsf{T}$ of sheaves S and T is defined by $(\mathsf{S} \times \mathsf{T})(U) = \mathsf{S}(U) \times \mathsf{T}(U)$, i.e., we just form the Cartesian product of section over each open set U. The analog of the one element set, the sheaf denoted 1, is defined by $1(U) = \{*\}$, i.e., has precisely one section over each open set. As in sets, this is also the product of no factors. The function space construction T^{S} is defined as follows. Let $\mathsf{S}|_U$ and $\mathsf{T}|_U$ denote the sheaves S and T restricted to the open set U [so $\mathsf{S}|_U$ is in Sh(U)]. Then for any open U, $\mathsf{T}^{\mathsf{S}}(U)$ is defined to be the set of sheaf maps from $\mathsf{S}|_U$ to $\mathsf{T}|_U$. This definition is in fact uniquely determined (cf. Johnstone, 1977) by the requirement that sheaf maps $\mathsf{S} \times \mathsf{R} \to \mathsf{T}$ be in one-to-one correspondence with sheaf maps $\mathsf{R} \to \mathsf{T}^{\mathsf{S}}$ (for all sheaves R). This requirement embodies the intended meaning of T^{S} as a function space. A sheaf S' is a subsheaf of a sheaf S (analog of subset) if $\mathsf{S}'(U) \subseteq \mathsf{S}(U)$ for all open sets U. The collection of all subsheaves of S is partially ordered by an inclusion relation $\leqslant$ defined by $\mathsf{S}' \leqslant \mathsf{S}''$ iff $\mathsf{S}'(U) \subseteq \mathsf{S}''(U)$ for all U. If S' and S'' are both subsheaves of S the intersection $\mathsf{S}' \cap \mathsf{S}''$ is defined by $(\mathsf{S}' \cap \mathsf{S}'')(U) = \mathsf{S}'(U) \cap \mathsf{S}''(U)$. Unions are more complicated since $U \mapsto \mathsf{S}'(U) \cup \mathsf{S}''(U)$ does not, in general, form a sheaf [it fails to satisfy the completeness condition (ii) of Definition 1.1]. Thus we define $\mathsf{S}' \cup \mathsf{S}''$ as the smallest subsheaf of S containing both S' and S''. An explicit description of $\mathsf{S}' \cup \mathsf{S}''$ is the following: $(\mathsf{S}' \cup \mathsf{S}'')(U)$ contains all those sections σ of S over U such that for some open cover U_i of U, $\rho_{UU_i}(\sigma)$ is either in $\mathsf{S}'(U_i)$ or $\mathsf{S}''(U_i)$, i.e., σ is locally in S' or S''.

In summary, the constructions of 1, products, subobjects, and intersections, are the same as those in sets, but carried out separately over each open set. The construction of function spaces and unions requires a notion of localization. We have not mentioned the restriction maps in the above construction since they are clear in each case.

Definition 3.1. The rules for the interpretation of terms and formulas written in a first-order many sorted language are shown in Table III.

The interpretation of sort constructions is as follows. Product sorts are interpreted by the corresponding sheaf products. $\{x|\varphi(x)\}$ for x of sort s is interpreted by the subsheaf $M(\varphi)$ of $M(s)$ and the sort $s_2^{s_1}$ is interpreted by $M(s_2)^{M(s_1)}$. Finally, the rule of satisfaction is that "M is a model of the formula $\varphi(x_1,\ldots,x_n)$" if $M(\varphi)\subseteq M(s_1)\times\cdots\times M(s_n)$ is the whole of $M(s_1)\times\cdots\times M(s_n)$.

The interpretation of a formula $\varphi(x_1,\ldots,x_n)$ is always a subsheaf of $M(s_1)\times\cdots\times M(s_n)$. Thus if φ has no free variables (i.e., is a sentence or proposition) then $M(\varphi)$ is a subsheaf of 1. Any subsheaf S of 1 has $\mathrm{S}(U)$ either $\{*\}$ or the empty set $\varnothing$, for each open $U\subseteq X$. Also if $\mathrm{S}(U)=\{*\}$ and $U'\subseteq U$ then $\mathrm{S}(U')$ must also be $\{*\}$ to make the restriction maps possible. The upshot of this is that S is completely characterized by the largest open set U over which $\mathrm{S}(U)=\{*\}$, i.e., there is a one-to-one correspondence between open subsets $U\subseteq X$ and subsheaves of 1. Hence we may regard sentences as being interpreted by open sets of X, i.e., the open sets play the role of the truth values. From this point of view, if φ and ψ are interpreted by the open sets U and V, respectively, then logical combinations are interpreted as follows:

$\varphi\&\psi$:	$U\cap V$
$\varphi\vee\psi$:	$U\cup V$
$\neg\varphi$:	interior (complement (U))
$\varphi\Rightarrow\psi$:	interior (complement $(U)\cup V$)

These operations with open sets correspond to the structure of a Heyting algebra (cf. Fourman and Scott, 1979). In contrast, in Section 2 the interpretation of sentences in sets formed a Boolean algebra under the logical operations. In fact the set of all subsets of a set always has a Boolean algebra structure, whereas the set of all subsheaves of a sheaf always has a natural Heyting algebra structure. The essential difference between Heyting and Boolean algebras is the behavior of negation. According to the above rules, the interpretations of $\neg\neg\varphi$ and φ in sheaf models are not, in general, the same. This shows that the logic of sheaves cannot be classical logic (classically $\neg\neg\varphi$ is equivalent to φ so their interpretations would have to be

TABLE III

Constructions in the language	Interpretation M in $\mathrm{Sh}(X)$
A sort s	$M(s)$ is a sheaf S
A relation symbol $R(x_1,\dots,x_n)$ (x_i of sort s_i)	A subsheaf $M(R)$ of the product sheaf $M(s_1)\times\cdots\times M(s_n)$
A function symbol $f(x_1,\dots,x_n)$ of sort s (x_i of sort s_i)	A sheaf map $M(s_1)\times\cdots\times M(s_n)\xrightarrow{M(f)} M(s)$
A constant c of sort s	A global section $M(c)$ of $M(s)$, i.e., a sheaf map $1\to M(s)$
A free variable x of sort s	The identity map $M(s)\to M(s)$
The inductivity for terms $t_1,\dots,t_n$ of sorts $s_1,\dots,s_n$ with free variables among sorts $s_1,\dots,s_j$:	
$f(t_1,\dots,t_n)$ of sort s	A sheaf map $M(s_1)\times\cdots\times M(s_j)\to M(s)$ obtained by composing the interpretations of the terms t_i with the interpretations of f
$R(t_1,\dots,t_n)$	$M(R(t_1,\dots,t_n))$ is the subsheaf of $M(s_1)\times\cdots M(s_j)$ obtained by forming the inverse image of $M(R)\subset M(s_1)\times\cdots\times M(s_n)$ under the map $M(t_1)\times\cdots\times M(t_n)$: $M(s_1)\times\cdots\times M(s_j)\to M(s_1)\times\cdots\times M(s_n)$
If all free variables of φ and ψ are among $x_1,\dots,x_n$ of sorts $s_1,\dots,s_n$ then:	
$\varphi\&\psi$	$M(\varphi\&\psi)=M(\varphi)\cap M(\psi)$ (intersection of subsheaves of the sheaf $M(s_1)\times\cdots\times M(s_n)$)
$\varphi\vee\psi$	$M(\varphi\vee\psi)=M(\varphi)\vee M(\psi)$
$\varphi\Rightarrow\psi$	$\sigma\in M(\varphi\Rightarrow\psi)(U)$ iff for any $U'\subseteq U$ if $\rho_{UU'}(\sigma)\in M(\varphi)(U')$ then $\rho_{UU'}(\sigma)\in M(\psi)(U')$
$\neg\varphi$	$\sigma\in M(\neg\varphi)(U)$ iff no restriction of σ belongs to any $M(\varphi)(U')$ $U'\subseteq U$
$\forall x_1\varphi(x_1,\dots,x_n)$	$\sigma\in(M(s_2)\times\cdots\times M(s_n))(U)$ is in $M(\forall x_1\varphi(x_1,\dots,x_n))$ iff for every open $U'\subseteq U$ and every $a'\in M(s_1)(U')$ we have the n-tuple $(a',\rho_{UU'}(\sigma))\in M(\varphi)(U')$
$\exists x_1\varphi(x_1,\dots,x_n)$	$\sigma\in(M(s_2)\times\cdots\times M(s_n))(U)$ is in $M(\exists x_1\varphi(x_1,\dots,x_n))$ iff there is an open cover U_i of U and elements $a_i\in M(s_1)(U_i)$ such that for each i the n-tuple $(a_i,\rho_{UU_i}(\sigma))\in M(\varphi)(U_i)$.

the same). It can be shown that sheaf categories respect the laws of *intuitionistic* logic (cf. Fourman and Scott, 1979) in the following sense: If M is a model of the formulas $\varphi_1,\ldots,\varphi_n$ and ψ is derivable from $\varphi_1,\ldots,\varphi_n$ using the laws of intuitionistic logic, then ψ is automatically also satisfied under the interpretation M. Models in the category of sets (and also sheaves over discrete topological spaces) have the extra property of respecting all the laws of *classical* logic. We will not list here the rules of intuitionistic logic (cf. Fourman and Scott, 1979, and Dummett, 1977) since they will not be required explicitly. Suffice it to say there these rules developed as a formalization of the purely constructivist foundation of mathematics originated by L. E. J. Brouwer. It is remarkable that the formalization of the notion of constructivity (in intuitionistic logic) and of the notion of continuity both lead to the same notion of topological space. This gives further reason to suspect that sheaf models may be useful in the modeling of the continuum.

The quantifier interpretation rules look simpler when stated for sentences. The sentence $\forall x\varphi(x)$ (x of sort s) is valid in a model M if $\varphi(a)$ holds for all local sections a in the sheaf $M(s)$, i.e., $M(\varphi)=M(s)$. $\exists x\varphi(x)$ is valid iff there is an open cover U_i of X with sections $\sigma_i\in M(s)(U_i)$ with $\sigma_i\in M(\varphi)(U_i)$, i.e., $\varphi(x)$ need not be witnessed by a global section of $M(s)$ but only locally on a cover of X. When computing interpretations of quantified sentences, it is often convenient to work with particular named *local* sections (rather than just global ones which can be interpreted as individual constants). Let $U\subseteq X$ be open. Any interpretation M in $\mathrm{Sh}(X)$ restricts to an interpretation M' in $\mathrm{Sh}(U)$ and a local section a of $M(s)$ over U becomes a global section of $M'(s)$ in $\mathrm{Sh}(U)$. We say that M satisfies $\varphi(a)$ in $\mathrm{Sh}(X)$ if M' satisfies $\varphi(a)$ in $\mathrm{Sh}(U)$ in the previously defined sense. Thus, for example, $\forall x\varphi(x)$ is valid in M iff M satisfies $\varphi(a)$ for all *local* sections a of $M(s)$.

4. EXAMPLES OF MATHEMATICAL CONSTRUCTIONS IN SHEAF MODELS

We describe how the rules of Section 3 are used to carry out some basic mathematical constructions in $\mathrm{Sh}(X)$ which we shall need later. As a first example, consider again the group axioms as stated in Section 2.

Proposition 4.1. In $\mathrm{Sh}(X)$ a sheaf G is a model of the group axioms iff G is a sheaf of groups (in the usual mathematical sense).

Proof. Let the sort G be interpreted by the sheaf $M(G)=\mathsf{G}$. Then μ and τ are interpreted as sheaf maps $M(\mu)$: $\mathsf{G}\times\mathsf{G}\to\mathsf{G}$, $M(\tau)$: $\mathsf{G}\to\mathsf{G}$, and $M(e)$

is a global section of G. Each of the three axioms is a sentence of the form $\forall x \varphi(x)$ (with x of sort $G \times G \times G, G, G$, respectively) and this is valid in M iff the formula $\varphi(x)$ has $M(\varphi) = \mathsf{G} \times \mathsf{G} \times \mathsf{G}, \mathsf{G}, \mathsf{G}$, respectively, i.e., for each open set U, $M(\mu)(U)$, $M(\tau)(U)$, and $M(e)$ restricted to U induce a group structure on $\mathsf{G}(U)$, i.e., G is a sheaf of groups. Conversely any sheaf of groups gives rise to an interpretation in which the axioms are satisfied.

Remark 4.2. In this example we have implicitly assumed that the equality relation on G is interpreted by $M(=) \subseteq \mathsf{G} \times \mathsf{G}$ having $M(=)(U) = \{(\sigma, \tau) | \sigma, \tau \in \mathsf{G}(U)$ and $\sigma = \tau\}$, i.e., equality is interpreted as meaning actual equality as sections, over any open set U. We will always use this interpretation of equality for all sorts even when not explicitly mentioned.

The second example concerns models of the axioms for a vector space. Let V be an n-dimensional (real) vector bundle over X. Let V be the sheaf of continuous sections of V and denote by K the sheaf of continuous real-valued functions on X.

Remark 4.3. K is a sheaf of rings and similar to Proposition 4.1, serves as a model for the ring axioms. [It is interesting to remark that the construction of the Dedekind reals from the natural numbers in Sh(X) gives K.] Let $\mathrm{Inv}(x)$ be an abbreviation for $\exists y(x.y = 1)$, i.e., $\mathrm{Inv}(x)$ is the predicate that x is invertible. The ring axioms can be extended to the axioms for an algebraic field by adding $\forall x(\neg(x = 0) \Rightarrow \mathrm{Inv}(x))$. A direct calculation shows that K does not satisfy this axiom[4] but it does satisfy the axiom $\forall x(\neg \mathrm{Inv}(x) \Rightarrow x = 0)$. These two axioms are classically equivalent but intuitionistically inequivalent (cf. Mulvey, 1974). Thus if we use the second form the axiom then K becomes an algebraic field in Sh(X). Note that with this definition, a field in Sh(X) is not a sheaf of fields.

For each open set U, $\mathsf{V}(U)$ is a module over the ring $\mathsf{K}(U)$ with respect to the usual pointwise operations of scalar multiplication and addition. It is straightforward to verify that V is a vector space over the field K. The local triviality of vector bundles combines with the local interpretation of the existential quantifier to yield the following result.

[4] This can be seen by the following illustrative case using $X = \mathbf{R}^n$ and interpreting sentences by open subsets of X. Let $U \subset X$ be open and f a real-valued function on U which vanishes only at an isolated point $x_0 \in U$. Working in Sh(U) the interpretation of $f = 0$ is the largest open subset of U on which f is zero, i.e., $\varnothing$ so the interpretation of $\neg f = 0$ is U. The interpretation U' of $\mathrm{Inv}(f)$ (using the local interpretation of $\exists$) is the union of all open subsets of U on which f is invertible, i.e., $U' = U - \{x_0\}$. Thus the interpretation of $\neg(f = 0) \Rightarrow \mathrm{Inv}(f)$ is interior [complement $(U) \cup U'] = U - \{x_0\}$. Since this is not the whole of U, the universally quantified statement cannot be satisfied since this requires the interpretation of $\neg(f = 0) \Rightarrow \mathrm{Inv}(f)$ to be U for all continuous sections f defined on all open sets U.

Proposition 4.4. Let V and K be as above. Let K be a sort subject to the field axioms and A a sort which is a vector space over K. Consider the sentence ("there is a basis of n elements generating A over K")[5]:

$$(\exists b_1,\ldots,b_n \in A)(\forall v \in A)(\exists! \alpha_1,\ldots,\alpha_n \in K)(v = \alpha_1 b_1 + \cdots + \alpha_n b_n) \tag{4.5}$$

Let M be an interpretation with $M(K) = \mathsf{K}$, $M(A) = \mathsf{V}$, with the obvious pointwise operations interpreting scalar multiplication and addition. Then the sentence (4.5) is valid under the interpretation M.

Proof. The interpretation of the sentence (4.5) is precisely the local triviality condition of vector bundles: $\exists b_1,\ldots,b_n$ in (4.5) requires the local existence over a cover U_i of X of n sections and the rest of (4.5) says simply that these n sections generate, by linear combinations, all the sections over U_i, in a unique way. All of this is guaranteed by the local structure of V being isomorphic to $U_i \times \mathbb{R}^n$ on a cover U_i.

Remark 4.6. The above result shows that vector bundles over X appear as vector spaces, in the intuitionistic set theory of Sh(X). An easy generalization shows that Proposition 4.4 remains true if X is a C^∞ manifold and K is the sheaf of smooth functions, and V is the sheaf of smooth sections of a smooth n-dimensional vector bundle; and also if X is a complex manifold and K and V are the analogous holomorphic constructions.

As a third example we develop some characteristics of sheaf models of topological spaces and continuous functions which will simplify computations in Section 6.

A model of a topological space in Sh(X) is a sheaf S with a collection F of subsheaves[6] (the "open subsheaves"). F is required to be closed under unions, finite intersections, and to contain S and the empty subsheaf. A sheaf map f: $\mathsf{S}_1 \to \mathsf{S}_2$ between models of topological spaces is continuous if the inverse image of every open subsheaf of S_2 is open in S_1.[6]

[5] $\exists! x\varphi(x)$ is read as "there exists a unique x such that $\varphi(x)$" and is an abbreviation for $\exists x \varphi(x)$ & $\forall x \forall y(\varphi(x)$ & $\varphi(y) \Rightarrow x = y)$. This formula is satisfied in an interpretation M iff there is an open cover U_i of X and unique sections σ_i over U_i with $\sigma_i \in M(\varphi)(U_i)$.

[6] A different formulation of topological spaces would require a *sheaf* of open subsheaves rather than just a collection. The formulation used here will be easier to handle computationally and sufficient for our purposes. In effect we are considering a topological space as a sort s with

An important class of topological spaces in Sh(X) is constructed from bundles over X. Let $Y \xrightarrow{p} X$ be a continuous surjection from a topological space Y to X. Denote by Y the sheaf of continuous sections of p. Y as an object in Sh(X) has a natural topological structure induced from Y: for each open $U \subseteq Y$ we have an open subsheaf $\mathsf{U} \subseteq \mathsf{Y}$ of all local sections of Y which take values in U. All open subsheaves of Y are of this form. We call this topology on Y the bundle topology. There is a simple characterization of continuous sheaf maps between bundle topologies:

> *Theorem 4.7.* (Fourman and Scott, 1979). Let S_1 and S_2 be bundles over X. Let S_2 be a T_0 space. Denote by S_1 and S_2 the sheaves of continuous sections of S_1 and S_2 endowed with the bundle topologies. Then there is a one-to-one correspondence between (i) continuous maps $\tilde{\varphi}$: $\mathsf{S}_1 \to \mathsf{S}_2$ [in Sh(X)] and (ii) continuous bundle maps φ: $S_1 \to S_2$ (in ordinary topological spaces). Given a bundle map φ, the corresponding sheaf map $\tilde{\varphi}$ is obtained by evaluating φ pointwise along sections, i.e., if $\sigma_1(x)$ is a section of S_1 over $U \subseteq X$ then $\tilde{\varphi}(\sigma_1)$ is the section of S_2 given by $\varphi(\sigma_1(x))$.

Suppose we wish to investigate in Sh(X), local properties of continuous S_2-valued maps on S_1. To each open subsheaf U of S_1 we associate the collection of continuous maps $\mathsf{U} \to \mathsf{S}_2$ and denote the totality of these sheaf maps by Cns($\mathsf{S}_1, \mathsf{S}_2$). According to the above theorem this object can be characterized by the sheaf A of S_2-valued bundle maps over S_1, in the sense that sections of A over $U \subset S_1$ are in one-to-one correspondence with continuous sheaf maps $\mathsf{U} \to \mathsf{S}_2$. Note that Cns($\mathsf{S}_1, \mathsf{S}_2$) is associated with the category Sh(X), whereas the simpler object A is an object in Sh(S_1). We will often use this representation and say that Cns($\mathsf{S}_1, \mathsf{S}_2$) is *represented as* A in Sh(S_1). In applications we shall use the following refinement of Theorem 4.7.

Proposition 4.8. Let S_1 and S_2 be complex analytic bundles over a complex manifold X. Let S_1 and S_2 be the sheaves of holomorphic sections endowed with the bundle topologies. Let $\mathsf{U} \subseteq \mathsf{S}_1$ be open. Then there is a one-to-one correspondence between (i) continuous sheaf maps $\tilde{\varphi}$: $\mathsf{U} \to \mathsf{S}_2$ and (ii) holomorphic bundle maps φ: $U \to S_2$. As in theorem 4.7, $\tilde{\varphi}$ is obtained from φ by pointwise evaluation along sections in U.

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infinitely many unary relations, one for each open subsort. This has, however, a technical drawback: since open sets do not collectively form a sort we cannot write statements in the language which quantify over open sets. Thus for example the condition of continuity cannot be written in the language unless we admit infinite disjunctions.

Proof. (In this proof U, V, etc. will denote open sets of S_1 or S_2 and U, V etc. will be the corresponding sheaves of sections taking values in these open sets.) Since any holomorphic bundle map is continuous, it follows easily from the definition of bundle topologies that the sheaf map $\tilde{\varphi}$ obtained by pointwise evaluation of φ along sections, is continuous. Conversely, given $\tilde{\varphi}$: $\mathsf{U} \to \mathsf{S}_2$, a continuous sheaf map, we construct an associated bundle map φ: $U \to S_2$.

We first show that if $\sigma_1(x)$ and $\sigma_2(x)$ are sections in U which intersect over $x_0 \in X$, i.e., $\sigma_1(x_0) = \sigma_2(x_0)$ in U then the continuity of $\tilde{\varphi}$ ensures that $\tilde{\varphi}(\sigma_1)$ and $\tilde{\varphi}(\sigma_2)$ also intersect over x_0. Thus, suppose that $a_1 = \tilde{\varphi}(\sigma_1)(x_0) \neq \tilde{\varphi}(\sigma_2)(x_0) = a_2$. Choose a neighborhood U_1 of a_1 not containing a_2. Then since $\tilde{\varphi}$ is continuous $\tilde{\varphi}^{-1}(\mathsf{U}_1)$ is of the form V_1 for some $V_1 \subset U$ open. Also $\sigma_1(x_0) \in V_1$ and since $\sigma_1(x_0) = \sigma_2(x_0)$, a restriction $\tilde{\sigma}_2$ of σ_2 is in V_1. Thus $\tilde{\varphi}(\tilde{\sigma}_2) \in \mathsf{U}_1$ so $a_2 \in U_1$ contradicting the construction of U_1. Thus $a_1 = a_2$. Now given $\tilde{\varphi}$: $\mathsf{U} \to \mathsf{S}_2$ construct φ as follows. Let $v \in U$ be over $x \in X$. We set

$$\varphi(v) = \tilde{\varphi}(\sigma)(x)$$

where σ is any section with $\sigma(x) = v$. This gives a well-defined bundle map independent of the choice of σ (by the above property of $\tilde{\varphi}$) and is easily seen to be continuous since $\tilde{\varphi}$ was. Furthermore the fact that $\tilde{\varphi}$ preserved all holomorphic sections leads to φ being a holomorphic bundle map. The constructions defined above for φ and $\tilde{\varphi}$ are readily seen to be inverses, giving the required correspondence.

As a final example we make some remarks on the interpretation in $\mathrm{Sh}(X)$, of the theory of complex analysis. This will be required since our theory of massless fields in the next section will be formulated for holomorphic fields on complexified space-time.

In mathematics it is usual to construct the complex numbers (and most other structures) out of the natural numbers or even the empty set. This extreme level of formalization seems to be of little value in the present work. In fact, it may be argued that from a physical point of view the ideas of discrete and continuous structures are essentially independent concepts. Accordingly, we shall seek axioms for a sort C to ensure a rich enough theory of complex analysis instead of constructing C as a completion of the real numbers. This problem, in the context of sheaf models, has been considered in detail by C. Rousseau (1979) and we quote some of her results. The axioms are stated relative to a sort R of real numbers. The sort C is required to have operations of addition and multiplication making it a field (respecting Remark 4.3). Furthermore C is equipped with a real-valued norm. The norm induces a topological structure on C in the usual way. Several other completeness and embedding properties relative to R are imposed, which we shall not state here since we do not use them explicitly. A sort C subject to these axioms is called a complex numbers sort. Rousseau shows that for a complex numbers sort we can define contour integration and that Cauchy's theorem and integration formula hold for interpretations in sheaf models. Holomorphic functions are defined in the standard way:

Definition 4.9. Let C be a complex numbers sort and U an open subsort. Then a map f: $U \to C$ is holomorphic with derivative g: $U \to C$ if the following sentence holds: $(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z' \in U)(|z - z'| < \delta \Rightarrow |f(z') - f(z) - g(z)(z - z')| < \varepsilon|z - z'|)$. (Here ε and δ are of sort R.) The basic result we shall use is the following.

Proposition 4.10. (Rousseau, 1979) Let X be a complex manifold. Let O be the sheaf of holomorphic functions on X. Then in $\mathrm{Sh}(X)$, O satisfies all the axioms of a complex numbers sort (with the obvious algebraic operations and norm; here R is interpreted by the sheaf of continuous real-valued functions on X). Furthermore the local holomorphic functions [in $\mathrm{Sh}(X)$] on O are represented on $X \times \mathbb{C}$ as the sheaf of holomorphic functions (in sets), i.e. For $\mathsf{U} \subseteq \mathsf{O}$ open in $\mathrm{Sh}(X)$, the holomorphic sheaf maps φ: $\mathsf{U} \to \mathsf{O}$ are exactly those obtained by evaluating holomorphic maps $\tilde{\varphi}$: $U \to \mathbb{C}$ along the sections of U.

Complex manifolds are defined in terms of a structure of holomorphic functions:

Definition 4.11. Let C be a complex numbers sort.

A complex manifold M of dimension n is a topological space M with a sheaf O_M of C-valued functions, called holomorphic functions satisfying the following: (a) If $f_1, \dots, f_k \in \mathsf{O}_M(U)$ and φ: $C^k \to C$ is a holomorphic function (in the sense of 4.9) then $\varphi(f_1, \dots, f_k) \in \mathsf{O}_M(U)$. (b) There is an open cover U_i of M and homeomorphisms μ_i: $U_i \to V_i$ with V_i open subsorts of C^n (called local coordinate systems) such that the holomorphic functions on U_i are isomorphic to the holomorphic functions on V_i via composition with μ_i.

Combining Propositions 4.8 and 4.10 gives the following.

Proposition 4.12. Let X be a complex manifold. Let the sheaf O of holomorphic functions on X be the interpretation of a complex numbers sort in $\mathrm{Sh}(X)$. Let Y be a complex manifold with $Y \xrightarrow{p} X$ a surjection whose fibers all have complex dimension n. Then the sheaf Y, of holomorphic sections of p is a model of an n-dimensional complex manifold in Sh(X). The sheaf of holomorphic functions on Y [in Sh(X)] is represented on Y as the sheaf of holomorphic functions on the complex manifold Y (in sets).

5. THE THEORY OF MASSLESS FIELDS

We set up a first-order many sorted language suitable for formulating the theory of analytic massless fields on complexified Minkowski space. In Section 6 we will be concerned with interpreting the theory in a sheaf category. The language and theory can be formulated in many (inequivalent) ways. Our choice is guided by the aim of relating the interpretational structures to the formalism of twistor theory, and follows Penroses spinor formulation of massless fields (Penrose 1968, to appear).

The language has four basic sort symbols denoted C, M, S_A, $S_{A'}$ which will denote, respectively, the domains of complex numbers, complex Minkowski space, unprimed spinors, and primed spinors. All other sorts that we use are constructed out of these. The sort C is subject to all of Rousseau's (1979) axioms for a complex numbers sort. M is required to be a topological space and a complex manifold (via Definition 4.11) relative to C. The local Lorentz structure of M will be expressed axiomatically later in terms of a relation between M and the spin spaces.

S_A and $S_{A'}$ are subject to the axioms of a vector space over C, with complex manifold structures making the algebraic operations holomorphic maps. The expression of $SL(2, C)$ action will be expressed in an unusual way, peculiar to sheaf models, and we leave the discussion of this until Remark 6.12. In effect the covariance property is partly built into the actual structure of the model and its logic, rather than being imposed axiomatically, so it appears at a more fundamental level than is usual. Also (related to this) we shall not require that both spin spaces be two dimensional, and later, allow the possibility that $S_{A'}$ be one dimensional. Consequently the skew two-index primed spinors usually used for index raising and lowering operations are no longer isomorphisms and we need to take more care with the axiomatics of primed spinor algebra manipulations [cf. Definition 1.3 (ii)]. The full spinor algebra is defined in terms of $C, S_A, S_{A'}$ in the usual way:

Definition 5.1. S^A and $S^{A'}$ are defined to be the C-linear duals of S_A and $S_{A'}$, respectively (these are defined as subsorts of the function space sorts C^{S_A} and $C^{S_{A'}}$). Spin space sorts with more than one index are defined as tensor products of the one-index sorts.

The local spinor structure of M is expressed as follows. We first axiomatize the sheaf Der of derivations of the holomorphic functions O_M on M, i.e., to each open subsort U of M we associate a sort of functions

$$\xi\colon \mathsf{O}_M(U) \to \mathsf{O}_M(U) \text{ satisfying}$$

$$(\forall f)(\forall g)(\xi(f+g) = \xi(f) + \xi(g))$$

$$(\forall f)(\forall g)(\xi(f\cdot g) = \xi(f)\cdot g + f\cdot \xi(g))$$

$$(\forall c \in C)(\forall f \in \mathsf{O}_M(U))(\xi(cf) = c\xi(f))$$

Also let the sort symbol $\mathsf{S}^{AA'}$ denote the sheaf of holomorphic $S^{AA'}$-valued functions on M. Then as part of the axiomatic structure of the sort M we shall require an O_M-linear injection

$$\Xi\colon \mathsf{S}^{AA'} \to \mathrm{Der}$$

This axioms differs from the one usually used in that Ξ is normally required to be an isomorphism. The stronger axiom also leads to some interesting models (cf. Jozsa, 1981) but the above weakening is necessary to allow models which are related to twistor theory in a simple way. Given the injection Ξ we can define a spinor covariant derivative (cf. Penrose, 1968).

Definition 5.2. A spinor covariant derivative is a map $\nabla_{AA'}\colon \mathsf{S}^{\cdots}_{\cdots} \hookrightarrow \mathsf{S}^{\cdots}_{\cdots AA'}$ with

(a) $\nabla(\varphi^{\cdots}_{\cdots} + \psi^{\cdots}_{\cdots}) = \nabla\varphi^{\cdots}_{\cdots} + \nabla\psi^{\cdots}_{\cdots}$.

(b) $\nabla(\varphi^{\cdots}_{\cdots}\psi^{\cdots}_{\cdots}) = \varphi^{\cdots}_{\cdots}\nabla\psi^{\cdots}_{\cdots} + (\nabla\varphi^{\cdots}_{\cdots})\psi^{\cdots}_{\cdots}$. (where there may be any number of contractions between $\varphi^{\cdots}_{\cdots}$ and $\psi^{\cdots}_{\cdots}$).

(c) $\nabla_{AA'}\nabla_{BB'}\varphi = \nabla_{BB'}\nabla_{AA'}\varphi$ for any holomorphic function φ.

(d) For any open subsort U of M and any $V^{AA'} \in \mathsf{S}^{AA'}(U)$, $\varphi \in \mathsf{O}_M(U)$ we have $\Xi(V^{AA'})\varphi = V^{AA'}(\nabla_{AA'}\varphi)$.

Given all this syntactical structure we now define massless fields and potentials exactly as in Definition 1.3. However we emphasize that in Section 1 this definition was referring to the usual model structures in sets. Here, the statements are regarded as purely syntactical sentences in our language and do not refer to any model theoretic structures.

Similarly we can transcribe into our language the two statements of Theorem 1.5. It can be shown that Theorem 1.5 is a valid logical consequence of the axioms we have set up in this section. Furthermore this logical deduction can be carried out using only the laws of intuitionistic logic (rather than full classical logic). The proof (cf. Jozsa, 1981) involves examining the (long) classical proof of this theorem and noting that, with minor modifications, all the steps are, in fact, intuitionistically valid. This is an important observation since the laws of intuitionistic deduction are valid in all sheaf models. Hence any sheaf model which satisfies the (relatively simple) axioms imposed on $C, M, S_A, S_{A'}$ will automatically be a model of the (relatively complicated) two statements in Theorem 1.5, i.e., the interpretations of these two statements in any such sheaf model will be guaranteed to yield sequences of sheaves, known to be exact, without any further work.

6. A SHEAF MODEL FOR THE THEORY OF MASSLESS FIELDS

It is clear that the theory of massless fields outlined in Section 5 has a standard model in the category of sets which reproduces all of the familiar structures of fields on space-time. In this section we shall develop a sheaf model of the theory in the category of sheaves over S^2 and show that it is related to the twistor description.

S^2 is the two real-dimensional sphere, and also possesses a unique structure of a complex manifold isomorphic to the complex projective line. In setting up the model it will be useful to think of this base space (for reasons given later) as the projective primed spin space of the usual set based model. To facilitate computations we introduce homogeneous coordinates $\pi_{A'} \in \mathbb{C}^2$ on the sphere S^2. The Lorentz covariance properties of this spinor will not be used in setting up the model and will enter only later when discussing the Lorentz covariance properties of the model. Thus, $\pi_{A'}$ for the time being is just a convenient way of referring to a pair of complex numbers.

We denote by O(0) the sheaf of holomorphic functions on S^2, and by $\mathrm{O}(n)$ (n an integer) the sheaf of holomorphic functions homogeneous of degree n. In terms of homogeneous coordinates, a section of $\mathrm{O}(n)$ is a local function $f(\pi_{A'})$ satisfying $\pi_{A'}(\partial/\partial\pi_{A'})f = nf(\pi_{A'})$. Alternatively, $\mathrm{O}(n)$ is the sheaf of holomorphic sections of the nth twisted line bundle on S^2, i.e., the line bundle of Chern class n. If $\mathfrak{S}^{\cdots}_{\cdots}$ is a spin space of the set based model then we denote by $\mathrm{O}^{\cdots}_{\cdots}(n)$ the sheaf on S^2 of holomorphic functions of degree n with values in $\mathfrak{S}^{\cdots}_{\cdots}$.

The model has the four basic sorts interpreted as shown in Table IV. All four sheaves in Table IV are sheaves of holomorphic sections of complex fiber bundles over S^2. According to Proposition 4.12 they therefore all have natural complex manifold structures in $\mathrm{Sh}(S^2)$. The algebraic operations on the interpretations of $C, S_A, S_{A'}$ are all taken to be the usual pointwise operations on sections (regarded as local functions on S^2) and according to

TABLE IV

Sort s	Interpretation $M(s)$ in $\mathrm{Sh}(S^2)$
C	$\mathrm{O}(0)$
S_A	$\mathrm{O}_A(0)$
$S_{A'}$	$\mathrm{O}(1)$
M	M, the sheaf of holomorphic sections of $CM \times S^2 \to S^2$, where CM is the usual set model of complexified Minkowski space

Remark 4.6 $\mathrm{O}_A(0)$ and $\mathrm{O}(1)$ are vector spaces over $\mathrm{O}(0)$ in $\mathrm{Sh}(S^2)$. By Proposition 4.10 $\mathrm{O}(0)$ satisfies the axioms for a complex numbers sort.

Remark 6.2. The interpretations of C, S_A, and M are all just the usual set based model structures parameterised holomorphically by the base space S^2, i.e., they are all sheaves of holomorphic sections of the product bundles $A \times S^2 \to S^2$, where A is the corresponding standard set based model. The interpretation of $S_{A'}$ is perhaps the most intriguing feature of the model and leads to its interesting properties. Note that according to Proposition 4.4 (and Remark 4.6) $\mathrm{O}_A(0)$ is two dimensional over $\mathrm{O}(0)$, whereas $\mathrm{O}(1)$ is only one dimensional. It is especially interesting that the base space S^2 can be thought of as supplying the "missing" dimension in the following way. According to the rules of interpretation the "completely defined elements" of a sheaf are the global sections. $\mathrm{O}(1)$ has a $\mathbb{C}^2$ family of global sections: in terms of our homogeneous coordinates on S^2 each global section is given explicitly by $f(\pi_{A'}) = \alpha^{A'}\pi_{A'}, \alpha^{A'} \in \mathbb{C}^2$. Each such section vanishes at a unique point of S^2 given by $\pi_{A'} = \alpha_{A'}$, and, given this point of S^2 the family of sections vanishing here is the projective equivalence class of global primed spinors $\{kf(\pi) = k\alpha^{A'}\pi_{A'}, k \in \mathbb{C}\}$. Thus we get a canonical identification of the points of the base space and the global projective primed spinors in the model. This peculiar way of representing part of the structure of the theory in the base space is not available in set models where the base space (the one-element set) is too small to carry any interesting structure.

We compute next the interpretations of spin spaces with arbitrary index structures.

Proposition 6.3. The dual spin spaces are given by $M(S^A) = \mathrm{O}^A(0)$ $M(S^{A'}) = \mathrm{O}(-1)$.

Proof. The dual spaces are constructed as spaces of C-linear functionals, regarded as subspaces of the full functions space sorts. From the interpretation of function space sorts in sheaf models given in Section 3 we see that the sections of $M(S^A)$ over $U \subseteq S^2$ are O(0) linear sheaf maps from $\mathrm{O}_A(0)|_U$ to $\mathrm{O}(0)|_U$. Since all algebraic operations are pointwise it readily follows that the only such linear maps are given by contraction with a section of $\mathrm{O}^A(0)$ over U, i.e., the sections of $M(S^A)$ over U are isomorphic to the sections of $\mathrm{O}^A(0)$ over U. Similarly we identify $M(S^{A'})$ as $\mathrm{O}(-1)$.

Proposition 6.4.

$$M(S_{A'B'}) = \mathrm{O}(2) = M\big(S_{(A'B')}\big)$$

$$M\big(S^{A'B'}\big) = \mathrm{O}(-2) = M\big(S^{(A'B')}\big)$$

$$M(S^{AB}) = \mathrm{O}^{AB}(0) \quad M(S_{AB}) = \mathrm{O}_{AB}(0)$$

$$M\big(S^{[A'B']}\big) = M\big(S_{[A'B']}\big) = 0$$

In general $M\left(S^{A_1\ldots A_n B'_1\ldots B'_p}_{C_1\ldots C_m D'_1\ldots D'_q}\right) = \mathrm{O}^{A_1\ldots A_n}_{C_1\ldots C_m}(q-p)$.

Proof. Spaces with more than one index are constructed as tensor products of the one-index spaces. We characterize tensor products axiomatically by the usual universality property with respect to C-multilinear maps (cf. MacLane and Birkhoff, 1967). This construction interprets in $\mathrm{Sh}(S^2)$ as the familiar tensor product of sheaves of modules over O(0) and all of the listed identifications follow immediately. The operation of outer multiplication is just the tensor product map in the above constructions, i.e., the pointwise outer multiplication of sections. The operation of contraction, defined by the action of spaces on their duals is also just multiplication of sections, with contractions over unprimed indices. Note also that since O(1) is one dimensional over O(0), tensor powers of $M(S_{A'})$ are automatically symmetrized and all spin spaces with skew primed indices vanish in the model.

Remark 6.5. In classical spinor algebra there is a theorem that any symmetric spinor is the symmetrized outer product of one index spinors. This theorem is, in fact equivalent to the fundamental theorem of algebra (Penrose and Rindler, to appear). It is interesting to note that this property does not necessarily hold in sheaf models as a result of the intuitionistic failure of the fundamental theorem of algebra. Thus, if we have a complex polynomial of degree n whose coefficients are parameterised holomorphically, by say, t, then although it can be solved to give n roots at each value of t we cannot in general construct the roots to vary holomorphically with t [cf. Jozsa (1981) for an explicit spinor example].

To identify the interpretation of the spinor derivative $\nabla_{AA'}$ we need to construct the interpretation of the derivations of holomorphic functions on M. Let F be the total space of the bundle $CM \times S^2 \to S^2$ (cf. Table IV entry for M). We introduce coordinates $x^{AA'}$ on CM so that F has coordinates $(x^{AA'}, \pi_{A'})$. According to Proposition 4.12 the interpretation of the sheaf of holomorphic functions on M in the category $\mathrm{Sh}(S^2)$ can be represented as the sheaf of holomorphic functions on F. We denote this sheaf by $\mathsf{F}(0)$. More generally we write $\mathsf{F}^{\cdots}_{\cdots}(n)$ for the sheaf on F of local holomorphic functions $f^{\cdots}_{\cdots}(x^{AA'}, \pi_{A'})$ taking values in the spin space $\mathcal{S}^{\cdots}_{\cdots}$ (of the model in sets) and being homogeneous of degree n in the $\pi_{A'}$ coordinate.

Proposition 6.6. The derivations of O_M are represented on F as the sheaf of maps of the form $v^{AA'}(x, \pi)(\partial/\partial x^{AA'})$: $\mathsf{F}(0) \to \mathsf{F}(0)$, where $v^{AA'}(x, \pi)$ is homogeneous of degree 0 in $\pi_{A'}$, i.e., the derivations are represented on F as $\mathsf{F}^{AA'}(0)$.

Proof. The representation of O_M as $\mathsf{F}(0)$ on F preserves addition and scalar multiplication of functions so derivations of O_M are represented on F as derivations of $\mathsf{F}(0)$. In coordinates, these all take the form

$$v^{AA'}(x, \pi)\frac{\partial}{\partial x^{AA'}} + f_{A'}(x, \pi)\frac{\partial}{\partial \pi_{A'}}.$$

However, the derivations are required to be $\mathsf{O}(0)$ linear in $\mathrm{Sh}(S^2)$. On F this amounts to the condition that the derivations vanish on $\mathsf{F}_\pi(0) \subseteq \mathsf{F}(0)$, the subsheaf of functions which are constant in $x^{AA'}$. This is equivalent to requiring that $f_{A'}$ be zero in the above expression. Finally the homogeneity of $v^{AA'}(x, \pi)$ is necessary to guarantee that the derivations take values in $\mathsf{F}(0)$.

In the model, the interpretation of $S^{AA'}$ fields on M, when represented on F, gives the sheaf $\mathsf{F}^A(-1)$. There is a natural injection

$$\mathsf{F}^A(-1) \to \mathsf{F}^{AA'}(0)$$

$$f^A(x, \pi) \mapsto f^A \pi^{A'}$$

which we use as the interpretation of the map Ξ appearing in Definition 5.2. The derivations obtained from $S^{AA'}$ fields via 5.2 (d) are all of the form

$$v^{AA'}(x, \pi)\frac{\partial}{\partial x^{AA'}} = f^A(x, \pi)\pi^{A'}\frac{\partial}{\partial x^{AA'}}$$

This identifies $M(\nabla_{AA'})$: $M(\mathsf{S}^{\cdots}_{\cdots}) \to M(\mathsf{S}^{\cdots}_{\cdots AA'})$ as the operator (when all function spaces are represented on F)

$$\pi^{A'}\frac{\partial}{\partial x^{AA'}} : \mathsf{F}^{\cdots}_{\cdots}(n) \to \mathsf{F}^{\cdots}_{\cdots A}(n+1)$$

Collecting together all the above constructions and remarks lead to the following.

Theorem 6.7. The interpretation given in Table IV with the subsequent definitions of analytic structure and the interpretations of Ξ and $\nabla_{AA'}$ provide a model for the theory of massless fields in the category $\mathrm{Sh}(S^2)$.

Next we construct the interpretations of massless fields themselves and of the sequences in Theorem 1.5.

Theorem 6.8. (a) A local function on M satisfying

$$\nabla_{AA'}\varphi^{A'\dots L'} = 0 \;(n \text{ symmetric indices } A'\dots L')$$

(i.e., a positive helicity massless field) interprets in the model, represented on F, as a local twistor function homogeneous of degree $-n$.

(b) A local function on M satisfying $\nabla_{X(X'}\varphi_{A'\dots L')} = 0$ (n symmetric indices $A'\dots L'$) interprets in the model, represented on F, as a local twistor function homogeneous of degree n.

Proof. For (a) we note that $S^{(A'\dots L')}$ valued fields are represented on F as $\mathsf{F}(-n)$ and the operation $\nabla_{AA'}$ with a primed contraction is simply the application of $\pi^{A'}\partial/\partial x^{AA'}$ so the interpretation of the field equation is

$$\pi^{A'}\frac{\partial}{\partial x^{AA'}} f(x,\pi) = 0$$

i.e., f is a twistor function. (b) is similar.

Let $\mathsf{T}(n) \subset \mathsf{F}(n)$ denote the sheaf on F of twistor functions homogeneous of degree n.

Theorem 6.9. The sequence in Theorem 1.5 (a) (regarded as a statement in the language of Section 5) interprets in the sheaf model (and represented on F) as the following sequence:

$$0 \to \mathsf{T}(n) \hookrightarrow \mathsf{F}(n) \xrightarrow{\pi^{A'}(\partial/\partial x^{AA'})} \mathsf{F}_A(n+1) \xrightarrow{\pi^{A'}(\partial/\partial x^{A'[B})} \mathsf{F}(n+2) \to 0,$$

$$n \leqslant -2$$

Proof. The interpretation of the first term is given in Theorem 6.8 (a) and the remaining three terms involving free holomorphic functions can easily be identified using Proposition 4.12. In the last term we have used the fact that $M(S_A)$ is two dimensional to identify $\mathsf{F}_{[AB]}(n)$ with $\mathsf{F}(n)$.

Theorem 6.10. The sequence in Theorem 1.5 (b) (regarded as a statement in the language of Section 5) interprets in the model as

$$0 \to \mathsf{T}(n) \hookrightarrow \mathsf{F}(n) \xrightarrow{\pi^{A'}(\partial/\partial x^{AA'})} \mathsf{G}_A(n+1) \to 0, \qquad n \geqslant 0$$

where $\mathsf{G}_A(n+1) \subseteq \mathsf{F}_A(n+1)$ is the subsheaf on F of functions satisfying

$$\pi^{A'} \nabla_{A'[A}\, g_{\,B]}(x, \pi) = 0$$

This sequence can be extended at the right-hand end to give the exact sequence

$$0 \to \mathsf{T}(n) \to \mathsf{F}(n) \xrightarrow{\pi^{A'}(\partial/\partial x^{AA'})} \mathsf{F}_A(n+1) \xrightarrow{\pi^{A'}(\partial/\partial x^{A'[B})} \mathsf{F}(n+2) \to 0$$

Proof. The identification of the terms $\mathsf{T}(n)$ and $\mathsf{F}(n)$ is similar to Theorems 6.9 and 6.8. The equation defining $\mathsf{G}_A(n+1) \subseteq \mathsf{F}_A(n+1)$ is easily verified to be the interpretation in the model (and represented on F) of the field equation for potentials in Definition 1.3 (iii). The final term in sequence 1.5 (b) involves skew primed spinors and is therefore zero in the model. The sequence can be extended in the designated way iff the map $\pi^{A'}(\partial/\partial x^{A'[B})$: $\mathsf{F}_A(n+1) \to \mathsf{F}(n+2)$ is surjective. This is, in fact, a true statement about sheaves on F, i.e., about structures in the model. Better still, surjectivity can be stated purely syntactically, in the language, and is exactly the interpretation in the model, of the statement that

$$\nabla_{A(A'} : \mathsf{S}^{A}_{(B' \ldots L')} \to \mathsf{S}_{(A' \ldots L')} \text{ is surjective} \tag{6.11}$$

It may be shown (cf. Jozsa, 1981) that if the spin spaces S^A, $S^{A'}$ are required to be either one or two dimensional (in the sense of Proposition 4.4) then the statement 6.11 is an intuitionistically valid deduction from the axioms of the theory. Thus any model satisfying these dimensionality axioms (e.g., our model) will automatically be a model of 6.11 and the required extension of the sequence follows immediately.

Theorems 6.8, 6.9, and 6.10 are the main results showing that the conventional spacetime description of massless fields and the basis of the twistor description are both obtained by interpreting a single theory of massless fields in two different sheaf categories [i.e., Sets and $\mathrm{Sh}(S^2)$].

The case $n = -1$ for the sequence 1.7:

$$0 \to \mathsf{T}(-1) \to \mathsf{F}(-1) \to \mathsf{F}_A(0) \to \mathsf{F}(1) \to 0$$

does not appear in the interpretations given by Theorems 6.9 and 6.10. This sequence arises as the interpretation in the model of the following sequence written in the language:

$$0 \to \mathsf{Z}^{A'} \hookrightarrow \mathsf{S}^{A'} \xrightarrow{\nabla_{AA'}} \mathsf{S}_A \xrightarrow{\nabla_{A'}{}^{\!}_{[B}} \mathsf{S}_{A'} \to 0$$

This sequence is not exact when $S_{A'}$ is two-dimensional. However, it is exact when $S_{A'}$ is one dimensional. Thus, since in our model, $M(S_{A'})$ is one dimensional, the exactness of this case, $n = -1$, follows again by model theoretic arguments.

Remark 6.12. We have left the discussion of Lorentz covariance properties of the model to this late stage since its proper expression requires a generalization of the definition 1.1 of sheaf, as outlined below. A naive approach would require an $SL(2, C)$ action pointwise along the sections of the sheaves interpreting the spin spaces. However, in view of Remark 6.2 this will not give a satisfactory group action on primed spinors which are partly represented in the base space. Evidently, the group action cannot be expressed by sheaf maps (since these always leave the base space fixed). Hence the group action is not expressed syntactically in the language but instead built, at a more fundamental level, into the structure of the model. These considerations motivate the following definitions.

Let X be a topological space and G a transitive group of homeomorphisms of X. For each $g \in G$ and $U \subseteq X$ open we have a map

$$g_U\colon U \to gU$$

with $gU \subseteq X$ open being the image of U under the action g. Let $\mathrm{Op}(X, G)$ be the category whose objects are the open sets of X and arrows are obtained by closure under composition of inclusions of open sets and the maps g_U for all $g \in G$, $U \subseteq X$ open. Note that any arrow $U \xrightarrow{\varphi} V$ in $\mathrm{Op}(X, G)$ can be decomposed as

$$U \xrightarrow{\varphi_1} W \xrightarrow{\varphi_2} V \qquad (2)$$

where W is the image of φ, φ_1 is of the form g_U for some $g \in G$, and φ_2 is the inclusion of the image of φ in V. Also, g_U: $U \to W$ is an isomorphism, with inverse h_W, where h is the inverse of g in G.

We define a cover of an open set U to be a family of arrows $\{\delta_i$: $U_i \to U\}$ of Op(X, G) which are jointly epimorphic, i.e., U is the union of the images of the δ_i. Given any such cover we can construct the pullback $U_j \xleftarrow{\varepsilon_j} U_{ij} \xrightarrow{\varepsilon_i} U_i$ of (δ_i, δ_j) as follows:

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varepsilon_i} & U_i \\ {\scriptstyle\varepsilon_j}\downarrow & & \downarrow{\scriptstyle\delta_i} \\ U_j & \xrightarrow[\delta_j]{} & U \end{array} \quad \text{commutes and}$$

$U_{ij} = \text{Image}(\delta_i) \cap \text{Image}(\delta_j)$. If according to (6.13) we decompose each δ_i as

$$U_i \xrightarrow{f_i} \text{Image}(\delta_i) \hookrightarrow U$$

and let k_i be the inverse of f_i, then the map ε_i is given by the composite

$$U_{ij} \hookrightarrow \text{Image}(\delta_i) \xrightarrow{k_i} U_i$$

This construction is (up to isomorphism) the natural analog in Op(X, G), of forming intersections of open sets $U_i, U_j \subseteq U$ in Op(X). [It is, in fact, an explicit description of the category theoretic pullback; cf. MacLane and Birkhoff (1967).

Definition 6.14

A G-equivariant sheaf on X is an assignment of a set $\mathrm{S}(U)$ to each object of Op(X, G) and for each arrow $U \xrightarrow{f} V$ in Op(X, G), a map $\mathrm{S}(V) \xrightarrow{\mathrm{S}(f)} \mathrm{S}(U)$ (a "generalized restriction map") satisfying the following.

(i) If

$$\begin{array}{ccc} U & \xrightarrow{h} & W \\ & {\scriptstyle f}\searrow \; V \; \nearrow{\scriptstyle g} & \end{array}$$

in Op (X, G) has $g \circ f = h$, then $\mathrm{S}(h) = \mathrm{S}(f) \circ \mathrm{S}(g)$: $\mathrm{S}(W) \to \mathrm{S}(U)$.

(ii) Let $\{\delta_i$: $U_i \to U\}$ be a cover of U and let $U_j \xleftarrow{\varepsilon_j} U_{ij} \xrightarrow{\varepsilon_i} U_i$ be the pullback of (δ_i, δ_j). Let $\sigma_i \in \mathsf{S}(U_i)$ be a collection of elements such that $\mathsf{S}(\varepsilon_i)(\sigma_i) = \mathsf{S}(\varepsilon_j)(\sigma_j)$ for all i, j. Then there is a unique $\sigma \in \mathsf{S}(U)$ with $\sigma_i = \mathsf{S}(\delta_i)(\sigma)$.

Note that Definition 6.14 parallels exactly Definition 1.1 except that the group G has been introduced to extend the notion of inclusions of open sets. The factorization (6.13) leads readily to a simpler description: it is sufficient that the completeness condition 6.14 (ii) hold only for coverings which do not include any group maps, i.e., only for the coverings of Definition 1.1.

Proposition 6.15. S is an equivariant sheaf over $\mathrm{Op}(X, G)$ iff (i) the sets $\mathsf{S}(U)$ form a sheaf over $\mathrm{Op}(X)$ (in the sense of Definition 1.1) and (ii) for each map $g_U\colon U \to V$ in $\mathrm{Op}(X, G)$ there is a map $\mathsf{S}(g_U)\colon \mathsf{S}(V) \to \mathsf{S}(U)$ satisfying all the composition conditions of Definition 6.14 (i).

Remark 6.16. It is shown in Johnstone (1977) and in Makkai and Reyes (1979) that the category of equivariant sheaves is rich enough to support model theoretic interpretation rules like those given in Section 3, which also respect intuitionistic logical deductions. In fact, both sheaves (Definition 1.1) and equivariant sheaves (Definition 6.14) are special cases of the more general notion of Grothendieck sheaves described at length in the above-mentioned references.

The notion of equivariant sheaves provides a natural solution to the problem of expressing the Lorentz covariance properties of our model.

The group $SL(2, C)$ acts on S^2 by homeomorphisms which are easily described in terms of the homogeneous coordinates $\pi_{A'} = (z_0, z_1)$: Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, C)$. Then the action is

$$(z_0, z_1) \mapsto (z_0, z_1)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

In the index notation we write the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ as $L_{A'}^{B'}$ and the action becomes

$$\pi_{A'} \mapsto L_{A'}^{B'}\pi_{B'}$$

Note that this map is homogeneous of degree one so composition with a homogeneous function of $\pi_{A'}$ preserves homogeneity.

All of the sheaves used to interpret the spin spaces have natural $SL(2, C)$ equivariance structures. $\mathsf{O}_A(0)$ and $\mathsf{O}(1)$ are made into $SL(2, C)$ equivariant sheaves (cf. Proposition 6.15) as follows. Let $U \subset S^2$ be open and $l \equiv L_{A'}^{B'} \in SL(2, C)$ so that $U \overset{l}{\text{---}} lU$ is an arrow in $\mathrm{Op}(S^2, SL(2, C))$. Let

$f(\pi_{A'}), \pi_{A'} \in lU$ be a section of O(1) over lU. Then the restriction of f along l is given by

$$f(\pi_{A'}) \mapsto f\left(L_{A'}^{B'}\sigma_{B'}\right), \qquad \sigma_{B'} \in U$$

Let $g_A(\pi_{A'})$ be a section of $O_A(0)$ over lU. The restriction of g_A along l is given by

$$g_A(\pi_{A'}) \mapsto \bar{L}_A^B g_B\left(L_{A'}^{B'}\sigma_{B'}\right), \qquad \sigma_{B'} \in U$$

(here $\bar{L}$ denotes the complex conjugate of L). This equivariance structure can be similarly defined for all the sheaves interpreting the spin space sorts. In fact we could have set up our entire model in the category of equivariant sheaves, at the expense of some transparency of presentation.

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